Tutorial 4 (7 Oct)

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Q1) Given a non-empty set A, define its space of bounded functions $\mathcal{B}(A)$ as

$$\mathcal{B}(A) = \{ f : A \to \mathbb{R} \mid f \text{ is bounded} : \exists M > 0 \text{ such that } \forall x \in A, |f(x)| \leq M \}$$

as a real vector space under pointwise addition and scalar multiplication.

Define the sup norm $\|\cdot\|_\infty : \mathcal{B}(A) \to \mathbb{R}$ as $\|f\|_\infty := \sup_{x \in A} \{ |f(x)| \}$.

Show that $(\mathcal{B}(A), \|\cdot\|_\infty)$ is a normed space.

Sol) Idea: Prove by the definition of normed space.

Recall the vector space structure of $\mathcal{B}(A)$, where

- pointwise addition: $\forall f, g \in \mathcal{B}(A)$, define $f+g : A \to \mathbb{R}$ as $(f+g)(x) = f(x) + g(x)$.
- scalar multiplication: $\forall \alpha \in \mathbb{R}, \forall f \in \mathcal{B}(A)$, define $\alpha f : A \to \mathbb{R}$ as $(\alpha f)(x) = \alpha \cdot f(x)$.

Note that if $M, N > 0$ satisfy $\forall x \in A$, $|f(x)| \leq M$ and $|g(x)| \leq N$, then

(i) $\forall x \in A$, $|(f+g)(x)| = |f(x) + g(x)| \leq |f(x)| + |g(x)| \leq M + N \Rightarrow f+g \in \mathcal{B}(A)$

(ii) $\forall x \in A$, $|(\alpha f)(x)| = |\alpha \cdot (f(x))| = |\alpha||f(x)| \leq |\alpha| \cdot M \Rightarrow \alpha f \in \mathcal{B}(A)$

Also, define constant zero function $f_0 : A \to \mathbb{R}$ as $f_0(x) = 0$. $(\Rightarrow f_0 \in \mathcal{B}(A))$

Exercise $(\mathcal{B}(A), +, \cdot)$ is a real vector space with f_0 as the zero vector.

Showing $(\mathbb{B}(A), \|\cdot\|_\infty)$ is a normed space:

$\forall f \in \mathbb{B}(A)$, $\|f\|_\infty := \sup_{x\in A}\{|f(x)|\}$ exists by completeness of $\mathbb{R}$.

(N1) $\forall f \in \mathbb{B}(A), \forall x \in A, |f(x)| \geq 0 \quad \therefore \|f\|_\infty \geq 0$,

Also, $\|f\|_\infty = 0 \iff \forall x \in A, |f(x)| = 0 \iff f = f_0$.

(N2) $\forall f \in \mathbb{B}(A), \forall \alpha \in \mathbb{R}$, showing $\|\alpha f\|_\infty = |\alpha|\|f\|_\infty$:

$[\leq]$ Note that $\forall x \in A, |f(x)| \leq \|f\|_\infty$. $\therefore$ By (ii), $\|\alpha f\|_\infty \leq |\alpha|\|f\|_\infty$.

$[\geq]$ $\forall x \in A$, $|\alpha||f(x)| = |\alpha\cdot(f(x))| = |(\alpha f)(x)| \leq \|\alpha f\|_\infty \quad \therefore \|\alpha f\|_\infty \geq |\alpha|\|f\|_\infty$.

(N3) $\forall f, g \in \mathbb{B}(A)$, showing $\|f+g\|_\infty \leq \|f\|_\infty + \|g\|_\infty$:

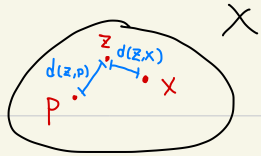
Note that $\forall x \in A, |f(x)| \leq \|f\|_\infty ; |g(x)| \leq \|g\|_\infty$. $\therefore$ By (i), $\|f+g\|_\infty \leq \|f\|_\infty + \|g\|_\infty$.

<u>Rmk</u> • This is a prototypical example of "function spaces with sup norm" inducing the following examples as normed spaces:

① $\mathcal{C}([a,b])$, the space of continuous functions on $[a,b]$. ($\Rightarrow$ bounded)

② $\mathcal{C}^b(X)$, the space of bounded continuous functions on a metric space (X,d).

③ $\ell_\infty = \mathbb{B}(\mathbb{N})$, the space of bounded sequences of real numbers.



Q2) Given a metric space (X, d) and a chosen point $p \in X$,

for each $x \in X$, define $f_x : X \to \mathbb{R}$ as $f_x(z) := d(z,x) - d(z,p)$.

(a) Show that f_x is bounded and Lipschitz continuous (hence $f_x \in \mathcal{C}^b(X)$).

(b) Define $\Phi : (X, d) \to (\mathcal{C}^b(X), d_\infty)$ as $\Phi(x) = f_x$.

Show that Φ is an isometric embedding :

$\forall x, x' \in X, \quad d_\infty(\Phi(x), \Phi(x')) := \| f_x - f_{x'} \|_\infty = d(x,x')$

(c) Hence, show that Φ is injective and continuous.

Sol) Idea : Apply the triangle inequalities of $(\mathbb{R}, |\cdot|)$ and (X, d) repeatedly.

(a) (i) Bounded : $\forall z \in X, \ |f_x(z)| = |d(z,x) - d(z,p)| \leq d(x,p)$.

(ii) Lipschitz Continuous : $\exists \ L > 0$ such that $\forall z, z' \in X, \ |f_x(z) - f_x(z')| \leq L \cdot d(z,z')$.

Note that $|f_x(z) - f_x(z')| = |(d(z,x) - d(z,p)) - (d(z',x) - d(z',p))|$

$\leq |d(z,x) - d(z',x)| + |d(z',p) - d(z,p)| \leq d(z,z') + d(z,z') = 2d(z,z')$. $\therefore$ Choose $L = 2$.

(b) $\forall x, x' \in X$, showing $\|f_x - f_{x'}\|_\infty = d(x,x')$:

[$\leq$]: $\forall z \in X$, $|f_x(z) - f_{x'}(z)| = |(d(z,x) - d(z,p)) - (d(z,x') - d(z,p))|$

$= |d(z,x) - d(z,x')| \leq d(x,x')$. $\therefore \|f_x - f_{x'}\|_\infty \leq d(x,x')$.

[$\geq$]: $\|f_x - f_{x'}\|_\infty \geq f_x(x') - f_{x'}(x') = d(x',x) - d(x',x') = d(x,x')$.

Hence, Φ is an isometric embedding.

(c) Showing Φ is injective: If $x, x' \in X$ such that $f_x = f_{x'}$,

then $d(x,x') = \|f_x - f_{x'}\| = 0 \Rightarrow x = x'$. Hence Φ is injective.

Showing Φ is continuous: $\forall x \in X$, $\forall \varepsilon > 0$, choose $\delta = \varepsilon > 0$,

then $\forall$ $x' \in X$ with $d(x,x') < \delta$, $\|f_x - f_{x'}\|_\infty = d(x,x') < \delta = \varepsilon$.

$\therefore$ Φ is continuous.

Rmk · Q2b.c imply X is "isometric isomorphic" to $\Phi(X) \subseteq (C^b(X), \|\cdot\|_\infty)$.

Hence, every metric space is a metric subspace of a certain function space.

· This provides an alternative way to construct a "completion" of X

as $\overline{\Phi(X)} \subseteq C^b(X)$, the closure of $\Phi(X)$ in $C^b(X)$.